

THE LOOP GROUP AND THE COBAR CONSTRUCTION

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ABSTRACT. We prove that for any 1-reduced simplicial set X , Adams' cobar construction ΩCX on the normalised chain complex of X is naturally a strong deformation retract of the normalised chains CGX on the Kan loop group GX . In order to prove this result, we extend the definition of the cobar construction and actually obtain the existence of such a strong deformation retract for all 0-reduced simplicial sets.

INTRODUCTION

There are two classical differential graded algebra models for the loop space on a 1-reduced simplicial set X : Adams' cobar construction ΩCX on the normalised chain complex [1], and the normalised chains CGX on the Kan loop group GX [7]. Both of these models are (weakly) equivalent to $C\Omega|X|$, the chains on the loop space of the realisation $|X|$.

In this article we show that ΩCX is actually a strong deformation retract of CGX , opening up the possibility of applying the tools of homological algebra to transferring perturbations of algebraic structure from the latter to the former.

Theorem. *For any 1-reduced simplicial set X there is a natural strong deformation retract of chain complexes*

$$(1) \quad \Omega CX \begin{array}{c} \xrightarrow{\phi} \\ \xleftarrow{\psi} \end{array} CGX \begin{array}{c} \curvearrowright \\ \Phi \end{array}$$

Here $\psi\phi$ is the identity map on ΩCX and Φ is a chain homotopy from $\phi\psi$ to the identity map on CGX . Furthermore both ϕ and ψ are homomorphisms of differential graded algebras.

In particular, ΩCX is isomorphic to a sub differential graded algebra of CGX , and both ϕ and ψ induce isomorphisms of algebras in homology.

Remark 1. *Let X and Y be 1-reduced simplicial sets, and $f : GX \rightarrow GY$ be a simplicial map (not necessarily a homomorphism). The theorem above gives us a natural way to construct a chain-level model of f . Indeed, if we set*

$$\xi = \psi \circ Cf \circ \phi : \Omega CX \rightarrow \Omega CY,$$

then

$$\begin{array}{ccc} \Omega CX & \xrightarrow{\xi} & \Omega CX \\ \phi \downarrow & & \downarrow \phi \\ CGX & \xrightarrow{Cf} & GCX \end{array}$$

commutes up to natural chain homotopy.

Remark 2. *It was proved in [6] that if X is a simplicial suspension, then ΩCX is naturally a primitively generated Hopf algebra, and the chain algebra map $\phi : \Omega CX \rightarrow CGX$ also respects comultiplicative structure. In this case the strong deformation retract of the theorem above is actually Eilenberg-Zilber data [3], which implies that the chain algebra map $\psi : CGX \rightarrow \Omega CX$ is also strongly homotopy comultiplicative [ibid.].*

In order to prove the theorem above, we extend the definition of the cobar construction and actually obtain the existence of such a strong deformation retract for all 0-reduced simplicial sets.

The homomorphism ϕ , which we recall in the first section of this article, was first described by Szczarba [11] in the language of twisting cochains. Given a simplicial set X that is 0-reduced but not necessarily 1-reduced, he gives an explicit, though somewhat complicated, formula for a twisting cochain,

$$\alpha_\phi : CX \rightarrow CGX,$$

that is based on the universal twisting function $\tau : X \rightarrow GX$ and that gives rise in usual way to an algebra homomorphism

$$\phi : \Omega CX \rightarrow CGX.$$

In degree zero the cobar construction is a free associative algebra on symbols given by the non-degenerate 1-simplices of X , while the right-hand side is the group ring on the free group on the same symbols. In the first section of the paper we observe that if X is not 1-reduced, then we may perform a change of rings along ϕ_0 , obtaining an extended cobar construction $\hat{\Omega}CX$, together with an algebra homomorphism

$$\phi : \hat{\Omega}CX \rightarrow CGX$$

that is an isomorphism in degree zero.

In the second section we introduce the retraction map ψ from the chains on the loop group to the extended cobar construction, for which we provide an explicit recursive formula. We prove in fact that ψ is a natural homomorphism of chain algebras and a one-sided inverse of the Szczarba map ϕ . It is surprising that such a map has not been previously observed in the literature.

In the third section we complete the strong deformation retraction (1) by defining the natural homotopy Φ . For this, we use the acyclic models for the loop group on 0-reduced simplices studied by Morace and Prouté [10].

1. PRELIMINARIES

1.1. Simplicial notions and notation. A simplicial set X is a contravariant functor from the category of finite non-empty ordinals Δ to the category of sets; more prosaically it is a sequence of sets X_n , $n \geq 0$, and specified face and degeneracy operators

$$d_i : X_n \rightarrow X_{n-1}, \quad s_i : X_n \rightarrow X_{n+1}, \quad (0 \leq i \leq n)$$

satisfying the simplicial identities, see for example [9]. A simplicial set is n -reduced if $X_k \cong \{*\}$ for $k \leq n$. The notions of simplicial group and simplicial objects in other categories are analogous.

Given an element $x \in X_n$ and any composite θ of simplicial face and degeneracy operators, represented by a monotonic function $f : \{0, \dots, m\} \rightarrow \{0, \dots, n\}$ in Δ , we also write

$$\theta(x) = x_{(f(0), \dots, f(m))}$$

for the corresponding element of X_m . We may write the face and degeneracy maps themselves, for example, as

$$\begin{aligned} d_i(x) &= x_{(0, \dots, \widehat{i}, \dots, n)}, \\ s_i(x) &= x_{(0, \dots, i, i, \dots, n)}. \end{aligned}$$

The simplicial relations imply that any simplicial operator $X_n \rightarrow X_m$ has normal form

$$\theta = s_{i_1} \dots s_{i_q} d_{j_1} \dots d_{j_r}$$

with $i_k > i_{k+1}$ and $j_k < j_{k+1}$ for all k . In this form, the corresponding *derived operator* $X_{n+1} \rightarrow X_{m+1}$ is

$$\theta' = s_{i_1+1} \dots s_{i_q+1} d_{j_1+1} \dots d_{j_r+1}.$$

An operator θ is *frontal* if it contains no d_0 ; such operators satisfy $\theta' s_0 = s_0 \theta$.

1.2. The cobar construction. We introduce a slightly extended definition of the cobar construction, which will be better suited for applying to the normalised chain complex on a 0-reduced simplicial set. Our definition generalises the classical construction of Adams, with which it agrees for simply connected coalgebras.

Let (C, ∂, Δ) be a connected differential graded coalgebra over a commutative ring R , so that $C_0 = R$. We suppose furthermore that C is R -free in each degree. Consider the ring B given by the free associative R -algebra on the desuspension of C_1 ,

$$B = T(s^{-1}C_1) = \bigoplus_{k \geq 0} (s^{-1}C_1)^{\otimes k},$$

Fix an R -basis $\{x_j \mid j \in J\}$ of C_1 , and let A be the ring obtained from B by freely adjoining inverses ξ_j of all elements of the form $1 + s^{-1}x_j$, for all $j \in J$. Explicitly,

$$A = T_B(\xi_j \mid j \in J) \quad / \quad (\xi_j \otimes (1 + s^{-1}x_j) = 1 = (1 + s^{-1}x_j) \otimes \xi_j).$$

Observe that the relations may also be expressed in the form

$$\xi_j \otimes s^{-1}x_j = 1 - \xi_j = s^{-1}x_j \otimes \xi_j.$$

The ring A may be regarded as a differential graded algebra concentrated in degree zero. The graded algebra underlying the extended cobar construction $(\hat{\Omega}C, \partial^\Omega)$ is then

$$\begin{aligned} \hat{\Omega}C &= T_A(s^{-1}C_{\geq 2}) \\ &= \bigoplus_{k \geq 0} A \otimes (s^{-1}C_{\geq 2} \otimes A)^{\otimes k}. \end{aligned}$$

Each R -module $(\hat{\Omega}C)_n$ is generated by words

$$a = a_1 \otimes \dots \otimes a_r, \quad |a_i| = n_i, \quad |a| = n = \sum n_i,$$

where either $a_i = s^{-1}c$ for some R -basis element $c \in C_{n_i+1}$ or $n_i = 0$ and $a_i = \xi_j$ for some $j \in J$. This R -module is free on those words in which ξ_j does not appear adjacent to $s^{-1}x_j$. The algebra multiplication is induced by concatenation of words, extended bilinearly to $\hat{\Omega}C$, modulo the relation that ξ_j is inverse to $1 + s^{-1}x_j$.

The differential ∂^Ω on $\hat{\Omega}C$ is the derivation of A -algebras that is specified by

$$\partial_n^\Omega s^{-1}c = -s^{-1}dc + (-1)^{|c|} s^{-1}c_i \otimes s^{-1}c^i$$

for all basis elements $c \in C_{n+1}$ and all $n \geq 1$, where $\Delta(c) = 1 \otimes c + c \otimes 1 + c_i \otimes c^i$ (using the Einstein summation convention). Note that ∂^Ω is necessarily zero on elements of A .

The unit $1 \in (\hat{\Omega}C)_0$ is identified with the empty word, via the isomorphism $R \cong (s^{-1}C_1)^{\otimes 0}$.

Remark 3. If C is simply connected, so that $C_1 = \{0\}$ and therefore $A = B = R$, then $\hat{\Omega}C$ coincides with the usual cobar construction ΩC defined by Adams.

1.3. The Kan loop group. Let X be a 0-reduced simplicial set and G a simplicial group. A *twisting function* $\tau : X \rightarrow G$ is a collection of functions of degree -1

$$\{\tau : X_{n+1} \rightarrow G_n \mid n \geq 0\}$$

satisfying

$$\begin{aligned} \tau s_0 x &= 1, \\ (2) \quad s_i \tau x &= \tau s_{i+1} x, \\ (3) \quad d_0 \tau x &= \tau d_0 x^{-1} \cdot \tau d_1 x, \\ (4) \quad d_i \tau x &= \tau d_{i+1} x \quad \text{if } i \geq 1. \end{aligned}$$

Let GX denote the Kan loop group on X , which is a simplicial group that models the space of based loops on the geometric realization of X (see [7, 9]). There is a *universal* twisting function

$$\begin{aligned} \tau : X_{n+1} &\rightarrow (GX)_n = \mathbf{F}(X_{n+1})/\mathbf{F}(s_0 X_n) \\ \tau x &= [x] \end{aligned}$$

sending $x \in X_{n+1}$ to the class of the corresponding generator in quotient of free groups, and the simplicial structure on GX is defined by (2)–(4).

Recall that the normalised chain complex CG on a simplicial group G also has a differential graded algebra structure, with multiplication given by the shuffle map and the multiplication in G ,

$$m : CG \otimes CG \longrightarrow C(G \times G) \longrightarrow CG,$$

that is,

$$(5) \quad m(g \otimes h) = \sum (-1)^{\text{sgn}(\mu, \nu)} s_{\mu(q)} \dots s_{\mu(1)} g \cdot s_{\nu(p)} \dots s_{\nu(1)} h, \quad g \in G_p, h \in G_q,$$

where the summation is over all (p, q) -shuffles $(\mu, \nu) \in \text{Shuff}(p, q)$.

The following proposition is the motivation for our extension of the cobar construction. Recall that for any simplicial set X , the degree n part of its normalised chain complex, $C_n X$, is the free abelian group on the set of all nondegenerate n -simplices of X and that CX has a comultiplication $\Delta : CX \rightarrow CX \otimes CX$ given by the Alexander-Whitney diagonal approximation

$$\Delta(x) = x_{(0)} \otimes x + \sum_{i=1}^{n-1} x_{(0, \dots, i)} \otimes x_{(i, \dots, n)} + x \otimes x_{(n)}$$

for $x \in X_n$, $n \geq 1$, and with $\Delta(x) = x \otimes x$ for $x \in X_0$. In particular, if X is 0-reduced, then CX is a connected, differential graded coalgebra over \mathbb{Z} .

Proposition 4. Let X be a 0-reduced simplicial set and GX its Kan loop group. Then there is an isomorphism of rings

$$(\hat{\Omega}CX)_0 \xrightleftharpoons[\psi_0]{\phi_0} (CGX)_0$$

determined by

$$\begin{aligned} \psi_0(\tau x) &= \xi_x & \psi_0(\tau x^{-1}) &= 1 + s^{-1}x \\ \phi_0(s^{-1}x) &= \tau x^{-1} - 1 & \phi_0(\xi_x) &= \tau x. \end{aligned}$$

Proof. The proof is straightforward. Note that if x is the degenerate element $s_0(*)$ then the four equations say $\psi_0(1) = 1$, $\phi_0(0) = 0$, $\phi_0(1) = 1$. In degree 0 the multiplication (5) is just $m(g \otimes h) = g \cdot h$, and we have

$$\begin{aligned}\psi_0(\tau x_1^{\alpha_1} \dots \tau x_r^{\alpha_r}) &= \psi_0(\tau x_1^{\alpha_1}) \otimes \dots \otimes \psi_0(\tau x_r^{\alpha_r}) \\ \phi_0(a_1 \otimes \dots \otimes a_r) &= \phi_0(a_1) \cdot \dots \cdot \phi_0(a_r)\end{aligned}$$

where $x_i \in X_1$, $\alpha_i = \pm 1$ and $a_i = s^{-1}x_i$ or ξ_{x_i} . This is well defined: $\psi_0(g)$ is inverse to $\psi_0(g^{-1})$ for all $g \in (GX)_0$ and $\phi_0(\xi_x)$ is inverse to $\phi_0(1 + s^{-1}x)$ for all $x \in X_1 \setminus \{s_0(*)\}$. It is also clear that the composites $\phi_0\psi_0$ and $\psi_0\phi_0$ are the respective identity maps:

$$\begin{aligned}\phi_0\psi_0(\tau x) &= \phi_0(\xi_x) = \tau x & \phi_0\psi_0(\tau x^{-1}) &= \phi_0(1 + s^{-1}x) = \tau x^{-1} \\ \psi_0\phi_0(s^{-1}x) &= \psi_0(\tau x^{-1} - 1) = s^{-1}x & \psi_0\phi_0(\xi_x) &= \psi_0(\tau x) = \xi_x.\end{aligned}$$

□

1.4. The Szczarba map. A map of differential graded algebras

$$\phi : \Omega CX \rightarrow CGX$$

was given explicitly by Szczarba in the language of twisting cochains. The following Definition, Lemma and Theorem are from sections 2 and 3 of Szczarba's paper [11], adapted slightly to define a map on the extended cobar construction

$$\phi : \hat{\Omega}CX \rightarrow CGX$$

that extends the isomorphism ϕ_0 of Proposition 4.

Let S_n be the set of $n!$ sequences of integers

$$i = (i_1, \dots, i_n) \quad \text{such that} \quad 0 \leq i_k \leq n - k \text{ for each } k.$$

In particular, $i_n = 0$. The *sign* of such a sequence $i \in S_n$ is

$$(-1)^{\sum i}, \quad \text{where} \quad \sum i = i_1 + i_2 + \dots + i_n.$$

Definition 5. Given a twisting function $\tau : X \rightarrow G$, the Szczarba operators are the functions

$$Sz_i : X_{n+1} \longrightarrow G_n, \quad i = (i_1, \dots, i_n) \in S_n,$$

given by the following product in G_n ,

$$Sz_i x = D_{0;i}^{n+1} \tau x^{-1} \cdot D_{1;i}^{n+1} \tau d_0 x^{-1} \cdot \dots \cdot D_{n;i}^{n+1} \tau d_0^n x^{-1}.$$

Here the operators $D_{j;i}^{n+1} : G_{n-j} \rightarrow G_n$ for $i \in S_n$, $j = 0, \dots, n$, are defined as

$$(6) \quad \begin{aligned} D_{0;()}^1 &= Id_{G_0}, \\ D_{j;i_1, \dots, i_n}^{n+1} &= \begin{cases} D_{j;i_2, \dots, i_n}^{n'} s_0 d_{i_1-j} & \text{if } j < i_1, \\ D_{j;i_2, \dots, i_n}^{n'} & \text{if } j = i_1, \\ D_{j-1;i_2, \dots, i_n}^{n'} s_0 & \text{if } j > i_1. \end{cases} \end{aligned}$$

As simplicial operators these are all frontal: defining $D_{j;i}^{n+1} : X_{n-j} \rightarrow X_n$ in the same way, one has $D_{j;i}^{n+1} \tau = \tau D_{j;i}^{n+1'} : X_{n-j+1} \rightarrow G_n$.

Lemma 6. The Szczarba operators satisfy

$$\begin{aligned} d_0 Sz_{i_1, \dots, i_n} &= Sz_{i_2, \dots, i_n} d_{i_1+1}, \\ d_k Sz_{i_1, \dots, i_k, i_{k+1}, \dots, i_n} &= d_k Sz_{i_1, \dots, i_{k+1}, i_k-1, \dots, i_n} \quad \text{if } i_k > i_{k+1}, \\ d_n Sz_{i_1, \dots, i_n} x &= s_\mu Sz_{i'} x_{(0, \dots, r)} \cdot s_\nu Sz_{i''} x_{(r, \dots, n+1)}.\end{aligned}$$

In the last equation the sequences i' , i'' , the integer r and the $(r-1, n-r)$ -shuffle (μ, ν) are defined by a certain bijection

$$\begin{aligned} S_n &\cong \bigcup_{r=1}^n \text{Shuff}(r-1, n-r) \times S_{r-1} \times S_{n-r} \\ i &\mapsto (\mu, \nu), \quad i', \quad i'' \end{aligned}$$

see [11, LEMMA 3.3], which respects parity as follows:

$$n + \sum i = r + \text{sgn}(\mu, \nu) + \sum i' + \sum i'' \pmod{2}.$$

Note that Szczarba's sign conventions differ slightly from ours, and that his inductively-defined parity $\epsilon(i, n+1)$ is in fact just $n + \sum i$.

Theorem 7. *For any twisting function $\tau : X \rightarrow G$ on a 0-reduced simplicial set X there is a canonical homomorphism of differential graded algebras*

$$\begin{aligned} \phi : \hat{\Omega}CX &\longrightarrow CG \\ \text{defined by} \quad \phi_0(\xi_{x_1}) &= \tau x_1 \\ \phi_0(s^{-1}x_1) &= \tau(x_1)^{-1} - 1 \\ \phi_n(s^{-1}x_{n+1}) &= \sum_{i \in S_n} (-1)^{\sum i} \text{Sz}_i x, \quad (n \geq 1), \end{aligned}$$

for $x_{n+1} \in X_{n+1}$.

Proof. The map ϕ extends linearly, and multiplicatively via

$$\phi_{p+q}(a \otimes b) = m(\phi_p(a) \otimes \phi_q(b)), \quad |a| = p, |b| = q,$$

where m is the multiplication (5), to all of $\hat{\Omega}CX$. We show ϕ is a chain map, i.e., that $\partial_n \phi_n = \phi_{n-1} \partial_n^\Omega$. For $x \in X_2$ we can write

$$\begin{aligned} \partial_1^\Omega s^{-1}x &= -s^{-1}d_0x + s^{-1}d_1x - s^{-1}d_2x - s^{-1}x_{(0,1)} \otimes s^{-1}x_{(1,2)} \\ &= (1 + s^{-1}d_1x) - (1 + s^{-1}x_{(0,1)}) \otimes (1 + s^{-1}x_{(1,2)}) \end{aligned}$$

and so we have, by Lemma 6,

$$\begin{aligned} \partial_1 \phi_1 s^{-1}x &= \partial_1 \text{Sz}_0 x = d_0 \text{Sz}_0 x - d_1 \text{Sz}_0 x = \text{Sz}_{()} d_1 x - \text{Sz}_{()} x_{(0,1)} \cdot \text{Sz}_{()} x_{(1,2)} \\ &= \tau(d_1 x)^{-1} - \tau(x_{(0,1)})^{-1} \cdot \tau(x_{(1,2)})^{-1} = \phi_0 \partial_1^\Omega s^{-1}x. \end{aligned}$$

For $x \in X_{n+1}$ the argument is essentially the same. We have

$$\begin{aligned} \partial_n \phi_n s^{-1}x &= \sum_{i \in S_n} (-1)^{\sum i} \partial_n \text{Sz}_i x \\ &= \sum_{i \in S_n} (-1)^{\sum i} \left(\sum_{k=0}^n (-1)^k d_k \text{Sz}_i x \right) \end{aligned}$$

where, by Lemma 6, all the terms for $0 < k < n$ cancel, and the terms for $k = 0, n$ may be rewritten as

$$\begin{aligned}
& \sum_{\substack{0 \leq i_1 \leq n-1 \\ i \in S_{n-1}}} (-1)^{i_1 + \sum i} S_{z_i} d_{i_1+1} x + \sum_{\substack{1 \leq r \leq n \\ (\mu, \nu), i', i''}} (-1)^{n + \sum i} s_\mu S_{z_{i'}} x_{(0, \dots, r)} \cdot s_\nu S_{z_{i''}} x_{(r, \dots, n+1)} \\
&= \sum_{1 \leq r \leq n} (-1)^{r-1} \phi_{n-1} s^{-1} d_r x + \sum_{1 \leq r \leq n} (-1)^r m(\phi_{r-1} s^{-1} x_{(0, \dots, r)} \otimes \phi_{n-r} s^{-1} x_{(r, \dots, n+1)}) \\
&\quad - 1 \otimes \phi_{n-1} s^{-1} x_{(1, \dots, n+1)} + (-1)^n \phi_{n-1} s^{-1} x_{(0, \dots, n)} \otimes 1 \\
&= \phi_{n-1} \left(\sum_{r=0}^{n+1} (-1)^{r-1} s^{-1} d_r x + \sum_{r=1}^n (-1)^r s^{-1} x_{(0, \dots, r)} \otimes s^{-1} x_{(r, \dots, n+1)} \right) \\
&= \phi_{n-1} \partial_n^\Omega x.
\end{aligned}$$

□

We will need one further property of the Szczarba operators.

Lemma 8. *For all $x \in X_{n+1}$ and $i \in S_n$, the following product in G_n is degenerate:*

$$D_{0; i}^{n+1} \tau x \cdot S_{z_i} x = D_{1; i}^{n+1} \tau d_0 x^{-1} \cdot \dots \cdot D_{n; i}^{n+1} \tau d_0^n x^{-1}.$$

Proof. For any sequence $i \in S_n$ we will show by induction that $D_{j; i}^{n+1}$ is $s_{\kappa(i)-1}$ -degenerate for all $j > 0$, where $\kappa(i)$ is the least integer such that $i_{\kappa(i)} = 0$. If $i_1 = 0$, so that $\kappa(i) = 1$, then by (6)

$$D_{j; i}^{n+1} = D_{j-1; i_2, \dots, i_n}^n s_0 = s_0 D_{j-1; i_2, \dots, i_n}^n$$

for all $j > i_1 = 0$. If $i_1 > 0$, so that $\kappa(i) > 1$, then $\kappa(i_2, \dots, i_n) = \kappa(i) - 1$ and we know that $D_{j; i_2, \dots, i_n}^n$ (if $j > 0$) and $D_{j-1; i_2, \dots, i_n}^n$ (if $j > 1$) are $s_{\kappa(i)-2}$ -degenerate by the inductive hypothesis. The corresponding derived operators are therefore $s_{\kappa(i)-1}$ -degenerate and, by (6), so is $D_{j; i}^{n+1}$ for all $j > 0$. □

2. THE RETRACTION MAP

2.1. Definition of the map. Let X be a 0-reduced simplicial set. We introduce in this section a map of differential graded algebras

$$\psi : CGX \longrightarrow \hat{\Omega}CX$$

between the chains on the loop group and the extended cobar construction, which is a retraction of the Szczarba map ϕ . The map ψ is uniquely determined by the relation

$$(7) \quad \psi_n(\tau x \cdot g) = \psi_n(g) - \sum_{i=0}^n s^{-1} x_{(0, \dots, i+1)} \otimes \psi_{n-i}(\tau d_1^i x \cdot d_0^i g)$$

for $x \in X_{n+1}$ and $g \in (GX)_n$. Note that the $i = 0$ term on the right-hand side is $s^{-1} x_{(0,1)} \otimes \psi_n(\tau x \cdot g)$. In fact ψ may be expressed inductively, on the degree n and the word length in $(GX)_n$.

Lemma 9. *The definition of ψ in (7) may be rewritten as*

$$\begin{aligned}
\psi_n(\tau x \cdot g) &= \xi_{x_{(0,1)}} \otimes \left(\psi_n(g) - \sum_{i=1}^n s^{-1} x_{(0, \dots, i+1)} \otimes \psi_{n-i} \omega_i(x, g) \right), \\
\psi_n(\tau x^{-1} \cdot h) &= (1 + s^{-1} x_{(0,1)}) \otimes \psi_n(h) + \sum_{i=1}^n s^{-1} x_{(0, \dots, i+1)} \otimes \psi_{n-i} \bar{\omega}_i(x, h).
\end{aligned}$$

where

$$\begin{aligned}\omega_i(x, g) &= \tau d_1^i x \cdot d_0^i g \in (GX)_{n-i}, \\ \bar{\omega}_i(x, h) &= \omega_i(x, \tau x^{-1} \cdot h) = \tau d_0 d_2^{i-1} x \cdot d_0^i h.\end{aligned}$$

Proof. Collecting the terms in (7) involving $\psi_n(\tau x \cdot g)$ and dividing by $1 + s^{-1}x_{(0,1)}$ gives the first equation. The second is obtained by taking $g = \tau x^{-1} \cdot h$ in the first. \square

From these formulae it is straightforward to give the map ψ explicitly in low degrees.

Lemma 10. *The map $\psi_0 : (GCX)_0 \rightarrow (\hat{\Omega}CX)_0$ agrees with that defined in Proposition 4, and the map $\psi_1 : (GCX)_1 \rightarrow (\hat{\Omega}CX)_1$ is given for $x, x_i \in X_2$ and $\alpha, \alpha_i = \pm 1$ by*

$$\begin{aligned}\psi_1(\tau x_1^{\alpha_1} \dots \tau x_r^{\alpha_r}) &= \sum_{i=1}^r \psi_0 d_1(\tau x_1^{\alpha_1} \dots \tau x_{i-1}^{\alpha_{i-1}}) \otimes \psi_1(\tau x_i^{\alpha_i}) \otimes \psi_0 d_0(\tau x_{i+1}^{\alpha_{i+1}} \dots \tau x_r^{\alpha_r}) \\ \text{with } \psi_1(\tau x^\alpha) &= \begin{cases} -\psi_0(\tau x_{(0,1)}) \otimes s^{-1}x \otimes \psi_0(\tau x_{(0,2)}) & (\alpha = +1) \\ s^{-1}x \otimes \psi_0(\tau x_{(1,2)}) & (\alpha = -1). \end{cases}\end{aligned}$$

Lemma 11. *ψ is well-defined.*

Proof. We show that $\psi_n(w) = 0$ if w is degenerate, by induction on n and on word length in GX . Suppose $0 \leq j \leq n-1$ and

$$w = s_j(\tau x^\alpha \cdot g),$$

where $\alpha = \pm 1$ and $\tau x^\alpha \cdot g$ is a reduced word in $(GX)_{n-1}$. For $\alpha = +1$ we have

$$\begin{aligned}(1 + s^{-1}x_{(0,1)}) \otimes \psi_n(w) &= (1 + s^{-1}x_{(0,1)}) \otimes \psi_n(\tau s_{j+1}x \cdot s_j g) \\ &= \psi_n s_j g - \sum_{i=1}^n (s_{j+1}x)_{(0, \dots, i+1)} \otimes \psi_{n-i} \omega_i(s_{j+1}x, s_j g)\end{aligned}$$

in which the first term is zero inductively. Each term in the summation is also zero since $(s_{j+1}x)_{(0, \dots, i+1)}$ is degenerate for $j < i$ and $\omega_i(s_{j+1}x, s_j g) = s_{j-i} \omega_i(x, g)$ for $j \geq i$. Since $1 + s^{-1}x_{(0,1)}$ is invertible, we have $\psi_n(w) = 0$.

The argument for $\alpha = -1$ is similar. \square

2.2. Properties of the retraction map. We now prove that ψ is a morphism of differential graded algebras and a retraction of the Szczarba map ϕ .

Proposition 12. *ψ is a chain map.*

Proof. We will show that for all $x \in X_{n+1}$ and $g \in (GX)_n$,

$$\partial_n^\Omega \psi_n(\tau x \cdot g) = \psi_{n-1} \partial_n(\tau x \cdot g),$$

by induction on n and on word length in GX . We first observe that

$$\begin{aligned}\psi_{n-1}(d_0(\tau x \cdot g) - \tau d_1 x \cdot d_0 g) &= \psi_{n-1}(\tau d_0 x^{-1} \cdot \tau d_1 x \cdot d_0 g) - \psi_{n-1}(\tau d_1 x \cdot d_0 g) \\ &= s^{-1}x_{(1,2)} \otimes \psi_{n-1}(\tau d_1 x \cdot d_0 g) + \sum_{i=1}^{n-1} s^{-1}x_{(1, \dots, i+2)} \otimes \psi_{n-1-i} \bar{\omega}_i(d_0 x, \tau d_1 x \cdot d_0 g),\end{aligned}$$

using the second formula in Lemma 9. Now since $\tau d_1 x \cdot d_0 g = \omega_1(x, g)$, and also $\bar{\omega}_i(d_0 x, \tau d_1 x \cdot d_0 g) = \bar{\omega}_i(d_1 x, \tau d_1 x \cdot d_0 g) = \omega_i(d_1 x, d_0 g) = \omega_{i+1}(x, g)$, we get

$$(8) \quad \psi_{n-1}(d_0(\tau x \cdot g) - \tau d_1 x \cdot d_0 g) = \sum_{k=1}^n s^{-1}x_{(1, \dots, k+1)} \otimes \psi_{n-k} \omega_k(x, g)$$

and, substituting $d_1^{r-1}x$ and $d_0^{r-1}g$ for x and g respectively, we obtain

$$(9) \quad \psi_{n-r}(d_0\omega_{r-1}(x, g) - \omega_r(x, g)) = \sum_{k=r}^n s^{-1}x_{(r, \dots, k+1)} \otimes \psi_{n-k}\omega_k(x, g).$$

Now, using Lemma 9, we know that

$$\begin{aligned} (1 + s^{-1}x_{(0,1)}) \otimes \partial_n^\Omega \psi_n(\tau x \cdot g) &= \partial_n^\Omega((1 + s^{-1}x_{(0,1)}) \otimes \psi_n(\tau x \cdot g)) \\ &= \partial_n^\Omega \left(- \sum_{i=1}^n s^{-1}x_{(0, \dots, i+1)} \otimes \psi_{n-i}\omega_i(x, g) + \psi_n g \right) \\ &= - \sum_{i=1}^n \partial_i^\Omega s^{-1}x_{(0, \dots, i+1)} \otimes \psi_{n-i}\omega_i(x, g) \\ &\quad - \sum_{i=1}^{n-1} (-1)^i s^{-1}x_{(0, \dots, i+1)} \otimes \psi_{n-1-i} \partial_{n-i}\omega_i(x, g) + \psi_{n-1} \partial_n g, \end{aligned}$$

since inductively $\partial_{n-i}^\Omega \psi_{n-i} = \psi_{n-i-1} \partial_{n-i}$ and $\partial_n^\Omega \psi_n g = \psi_{n-1} \partial_n g$. Now expanding the operators ∂_n^Ω and ∂ we get

$$\begin{aligned} (1 + s^{-1}x_{(0,1)}) \otimes \partial_n^\Omega \psi_n(\tau x \cdot g) &= \\ &- \sum_{i=1}^n \sum_{r=0}^{i+1} (-1)^{r+1} s^{-1}x_{(0, \dots, \hat{r}, \dots, i+1)} \otimes \psi_{n-i}\omega_i(x, g) \\ (10) \quad &- \sum_{i=1}^n \sum_{q=1}^i (-1)^q s^{-1}x_{(0, \dots, q)} \otimes s^{-1}x_{(q, \dots, i+1)} \otimes \psi_{n-i}\omega_i(x, g) \\ &- \sum_{j=1}^{n-1} \sum_{t=0}^{n-j} (-1)^{j+t} s^{-1}x_{(0, \dots, j+1)} \otimes \psi_{n-1-j} d_t \omega_j(x, g) + \sum_{k=0}^n (-1)^k \psi_{n-1} d_k g. \end{aligned}$$

Collecting together the terms for which either $i = 1$, $k = 0$, $r = 0$, or $q = 1$ gives

$$\begin{aligned} &\left((1 + s^{-1}x_{(0,1)}) - (1 + s^{-1}x_{(0,2)}) \right) \otimes \psi_{n-1}\omega_1(x, g) + \psi_{n-1}d_0g \\ &+ \sum_{i=1}^n (1 + s^{-1}x_{(0,1)}) \otimes s^{-1}x_{(1, \dots, i+1)} \otimes \psi_{n-i}\omega_i(x, g) \\ &= (1 + s^{-1}x_{(0,1)}) \otimes \psi_{n-1}d_0(\tau x \cdot g) - (1 + s^{-1}x_{(0,2)}) \otimes \psi_{n-1}\omega_1(x, g) + \psi_{n-1}d_0g \end{aligned}$$

by (8), and by Lemma 9 the last two terms here cancel exactly with the terms for $r = 1$ and $i > 1$ in (10).

Now collecting the terms for $r = i + 1$ and $i > 1$ in (10), together with all the (i, q) -indexed terms not already considered, gives

$$\begin{aligned} &- \sum_{i=2}^n \left((-1)^i s^{-1}x_{(0, \dots, i)} + \sum_{q=2}^i (-1)^q s^{-1}x_{(0, \dots, q)} \otimes s^{-1}x_{(q, \dots, i+1)} \right) \otimes \psi_{n-i}\omega_i(x, g) \\ &= - \sum_{r=2}^n (-1)^r s^{-1}x_{(0, \dots, r)} \otimes \left(\psi_{n-r}\omega_r(x, g) + \sum_{k=r}^n s^{-1}x_{(r, \dots, i+1)} \otimes \psi_{n-i}\omega_i(x, g) \right) \\ &= - \sum_{r=2}^n (-1)^r s^{-1}x_{(0, \dots, r)} \otimes \psi_{n-r}d_0\omega_{r-1}(x, g) \end{aligned}$$

by (9), and this cancels exactly with the terms for $t = 0$ in (10).

Thus expression (10) is equal to

$$\begin{aligned}
(11) \quad & (1 + s^{-1}x_{(0,1)}) \otimes \psi_{n-1}d_0(\tau x \cdot g) + \sum_{k=1}^n (-1)^k \psi_{n-1}d_k g \\
& - \sum_{2 \leq r \leq i \leq n} (-1)^{r+1} s^{-1}x_{(0, \dots, \widehat{r}, \dots, i+1)} \otimes \psi_{n-i}\omega_i(x, g) \\
& - \sum_{1 \leq j < j+t \leq n} (-1)^{j+t} s^{-1}x_{(0, \dots, j+1)} \otimes \psi_{n-1-j}d_t \omega_j(x, g).
\end{aligned}$$

Now to complete the proof it remains to show that this is equal to

$$\begin{aligned}
(1 + s^{-1}x_{(0,1)}) \otimes \psi_{n-1}\partial_n(\tau x \cdot g) &= (1 + s^{-1}x_{(0,1)}) \otimes \psi_{n-1}d_0(\tau x \cdot g) \\
&+ \sum_{i=1}^n (-1)^i (1 + s^{-1}x_{(0,1)}) \otimes \psi_{n-1}(\tau d_{i+1}x \cdot d_i g).
\end{aligned}$$

The first term is as required, and by Lemma 9 the summation is

$$\sum_{i=1}^n (-1)^i \left(\psi_{n-1}d_i g - \sum_{k=1}^{n-1} s^{-1}(d_{i+1}x)_{(0, \dots, k+1)} \otimes \psi_{n-1-k}\omega_k(d_{i+1}x, d_i g) \right).$$

The result therefore follows from the observations that

$$(d_{i+1}x)_{(0, \dots, k+1)} = \begin{cases} x_{(0, \dots, \widehat{i+1}, \dots, k+2)} & \omega_k(d_{i+1}x, d_i g) = \begin{cases} \omega_{k+1}(x, g) & (i \leq k), \\ d_{i-k}\omega_k(x, g) & (i > k). \end{cases} \end{cases}$$

□

Proposition 13. *The map ψ is an algebra homomorphism.*

Proof. Let $v \in (GX)_p$ and $w \in (GX)_q$ and consider $v \otimes w \in (CGX \otimes CGX)_n$, $n = p + q$. To show that ψ is multiplicative we must prove

$$\psi_n m(v \otimes w) = \sum_{(\mu, \nu)} (-1)^{\text{sgn}(\mu, \nu)} \psi_n(s_\mu v \cdot s_\nu w) = \psi_p v \otimes \psi_q w$$

in $\hat{\Omega}CX$, by induction on p and the word length of v . For $v = *$, or p or $q = 0$, there is nothing to prove; suppose inductively that $p, q \geq 1$ and $v = \tau x \cdot g$ for $x \in X_{p+1}$ (the argument for $v = \tau x^{-1} \cdot g$ is similar). Then by Lemma 9,

$$\begin{aligned}
(1 + s^{-1}x_{(0,1)}) \otimes \psi_n(s_\mu(\tau x \cdot g) \cdot s_\nu w) &= (1 + s^{-1}x_{(0,1)}) \otimes \psi_n(\tau s'_\mu x \cdot (s_\mu g \cdot s_\nu w)) \\
&= \psi_n(s_\mu g \cdot s_\nu w) - \sum_{i=1}^n d_{i+2}^{n-i} s'_\mu x \otimes \psi_{n-i}\omega_i(s'_\mu x, s_\mu g \cdot s_\nu w)
\end{aligned}$$

The term $d_{i+2}^{n-i} s'_\mu x$ will be degenerate unless $i \leq p$ and (s_μ, s_ν) is of the form $(s_{i+\xi_q} s_{i+\xi_{q-1}} \dots s_{i+\xi_1}, s_0^i s_\zeta)$ for some $(p-i, q)$ -shuffle (ξ, ζ) , and we have

$$\begin{aligned}
& (1 + s^{-1}x_{(0,1)}) \otimes \psi_n m(v \otimes w) \\
&= \sum_{(\mu, \nu)} (-1)^{\text{sgn}(\mu, \nu)} \psi_n(s_\mu g \cdot s_\nu w) - \sum_{\substack{1 \leq i \leq p \\ (\xi, \zeta)}} (-1)^{\text{sgn}(\xi, \zeta)} d_{i+2}^{p-i} x \otimes \psi_{n-i}(s_\xi \omega_i(x, g) \cdot s_\zeta w) \\
&= \psi_n m(g \otimes w) - \sum_{i=1}^p d_{i+2}^{p-i} x \otimes \psi_{n-i} m(\omega_i(x, g) \otimes w) \\
&= \left(\psi_p g - \sum_{i=1}^p d_{i+2}^{p-i} x \otimes \psi_{p-i} \omega_i(x, g) \right) \otimes \psi_q w, \quad \text{by the inductive hypothesis,} \\
&= (1 + s^{-1}x_{(0,1)}) \otimes \psi_p v \otimes \psi_q w,
\end{aligned}$$

by Lemma 9. The result follows. \square

Proposition 14. *The map ψ is a retraction of ϕ , that is, the composite $\psi\phi$ is the identity.*

Proof. It is enough to prove this on algebra generators of $\hat{\Omega}CX$. For $x \in X_{n+1}$ and $i = (i_1, \dots, i_n) \in S_n$ we will show that

$$\psi_n \text{Sz}_i x = \begin{cases} x & \text{if } i_1 = \dots = i_n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Denote by $x_{0;i}$ the element of X satisfying

$$D_{0;i}^{n+1} \tau x = \tau x_{0;i}.$$

Lemma 8 tells us $\psi_n(\tau x_{0;i} \cdot \text{Sz}_i x) = 0$, and so by Lemma 9 we have

$$\psi_n \text{Sz}_i x = \sum_{k=1}^n d_{k+2}^{n-k} x_{0;i} \otimes \psi_{n-k} \omega_k(x_{0;i}, \text{Sz}_i x).$$

From (6) we see that $D_{0,i}^{n+1} \tau x$ has an s_{k-1} degeneracy if $i_k \neq 0$. Thus $d_{k+2}^{n-k} x_{0;i}$ is degenerate except in the case $i_1 = \dots = i_k = 0$. In this case we see from (6) and Lemma 6 that

$$\omega_k(x_{0;i}, \text{Sz}_i x) = \tau d_1^k x_{0;i} \cdot d_0^i \text{Sz}_i x = D_{0;i_{k+1}, \dots, i_n}^{n-k+1} \tau d_1^k x \cdot \text{Sz}_{i_{k+1}, \dots, i_n} d_1^k x$$

which is degenerate again by Lemma 8. The only non-zero term is therefore

$$\psi_n \text{Sz}_{0, \dots, 0} x = x_{0;0, \dots, 0} \otimes \psi_0(*) = x,$$

and hence $\psi\phi x = x$ as required. \square

3. DEFORMATION RETRACTION OF THE LOOP GROUP

Both the Kan functor G and the cobar construction model loop spaces. In the 1-reduced case it is easy to show that the Szczarba map $\phi : \Omega CX \rightarrow CGX$ is a weak equivalence, by applying Zeeman's comparison theorem to the map of spectral sequences associated with

$$(12) \quad \begin{array}{ccccc} \Omega CX & \longrightarrow & \Omega CX \otimes_{\alpha_\phi} CX & \longrightarrow & CX \\ \phi \downarrow & & \downarrow & & \downarrow = \\ CGX & \longrightarrow & C(GX \times_\tau X) & \longrightarrow & CX \end{array}$$

in which the total spaces are acyclic.

We prove here the following stronger result.

Theorem 15. *Let X be a 0-reduced simplicial set. Let ϕ be the Szczarba map and ψ the retraction map defined above.*

There is a natural strong deformation retraction of chain complexes

$$\hat{\Omega}CX \begin{array}{c} \xrightarrow{\phi} \\ \xleftarrow{\psi} \end{array} CGX \begin{array}{c} \curvearrowright \\ \Phi \end{array}$$

Recall that if A and B are chain complexes, $\nabla : A \rightarrow B$ and $f : B \rightarrow A$ are chain maps, and $h : B \rightarrow B$ is linear map of degree $+1$, then

$$A \begin{array}{c} \xrightarrow{\nabla} \\ \xleftarrow{f} \end{array} B \begin{array}{c} \curvearrowright \\ h \end{array}$$

is a strong deformation retract if $f\nabla = Id_A$ and $\partial h + h\partial = \nabla f - Id_B$. Given a strong deformation retract, one can apply the machinery of homological perturbation theory to transfer perturbations of the structure B across to A , obtaining a new strong deformation retract [4], [5], [8].

Proof. According to Proposition 14, we need only to prove that there is a natural chain homotopy from the composite $\phi\psi$ to the identity map on CGX . The proof, an acyclic models argument, proceeds by induction on degree.

The base step of the induction is trivial, by Proposition 4. We can simply set $\Phi_0 = 0 : C_0GX \rightarrow C_1GX$ for all 0-reduced simplicial sets X .

Suppose now that $\Phi_k : C_kGX \rightarrow C_{k+1}GX$ has been defined for all $0 \leq k < n$ and for all 0-reduced simplicial sets X so that

- (1.k) Φ_k is natural in X for all k , and
- (2.k) $\partial\Phi_k + \Phi_{k-1}\partial = \phi\psi - Id_{CGX}$ for all k and all X ,

where $n \geq 1$.

Let $\overline{\Delta}[n]$ denote the quotient of the standard simplicial n -simplex by its 0-skeleton. If $x = (k_0 \cdots k_j)$ is a j -simplex of $\Delta[n]$, let $x \cdot n$ denote the $(j+1)$ -simplex $(k_0 \cdots k_j n)$. Let

$$h_i^n : (G\overline{\Delta}[n])_i \rightarrow (G\overline{\Delta}[n])_{i+1}$$

denote the group homomorphism specified by $h_i^n(\tau x) = \tau(x \cdot n)$ for all $x \in \overline{\Delta}[n]_{i+1}$. Let

$$\bar{h}^n : C_*G\overline{\Delta}[n] \rightarrow C_{*+1}G\overline{\Delta}[n]$$

denote the degree +1 linear map specified by $\bar{h}_i^n(w) = -h_i^n(w)$ for all $w \in (G\overline{\Delta}[n])_i$.

Morace and Prouté proved in [10] that for all $i \geq 1$

$$\partial_{i+1}\bar{h}_i^n + \bar{h}_{i-1}^n\partial_i = Id,$$

i.e., that \bar{h}^n is a contraction in positive degrees. It follows that $H_iG\overline{\Delta}[n] = 0$ for all $i \geq 1$.

Consider the infinite wedge

$$W(n+1) = \bigvee_{m \in \mathbb{N}} \overline{\Delta}[n+1].$$

There is a chain homotopy

$$\tilde{h}^{n+1} : C_*GW(n+1) \rightarrow C_{*+1}GW(n+1)$$

that is a contraction in positive degrees and that generalizes Morace and Prouté's construction.

Let

$$w = \tau\delta_{m_1}^{\alpha_1} \cdots \tau\delta_{m_k}^{\alpha_k} \in (GW(n+1))_n,$$

where δ_{m_i} denotes the unique nondegenerate $(n+1)$ -simplex in the m_i^{th} copy of $\overline{\Delta}[n+1]$ in $W(n)$, and $\alpha_i = \pm 1$. Set

$$\Phi_n(w) = \tilde{h}^n(\phi\psi(w) - w - \Phi_{n-1}(\partial w)).$$

The induction hypothesis implies that $\phi\psi(w) - w - \Phi_{n-1}(\partial w)$ is a cycle and that

$$\begin{aligned} \partial\Phi_n(w) &= -\tilde{h}^n\partial(\phi\psi(w) - w - \Phi_{n-1}(\partial w)) + \phi\psi(w) - w - \Phi_{n-1}(\partial w) \\ &= \phi\psi(w) - w - \Phi_{n-1}(\partial w). \end{aligned}$$

Adding $\Phi_{n-1}(\partial w)$ to both sides of this equation, we obtain

$$\partial\Phi_n(w) + \Phi_{n-1}(\partial w) = \phi\psi(w) - w,$$

i.e., (2.n) holds for all such w .

Let X be any 0-reduced simplicial set, and let

$$w = \tau x_1^{\alpha_1} \cdot \dots \cdot \tau x_k^{\alpha_k}$$

be any nondegenerate n -simplex in GX , where $\alpha_i = \pm 1$ for all i . Let $\zeta_i : \overline{\Delta}[n+1] \rightarrow X$ be the simplicial map representing x_i .

Let $\gamma : \overline{\Delta}[n+1] \rightarrow X$ denote the simplicial map collapsing everything to the basepoint. Consider the morphism of simplicial groups

$$\Psi_w = G(\zeta_1 \vee \dots \vee \zeta_k \vee \bigvee_{m>k} \gamma) : GW(n+1) \rightarrow GX.$$

Observe that

$$\Psi_w(\tau \delta_1^{\alpha_1} \cdot \dots \cdot \tau \delta_k^{\alpha_k}) = w,$$

where δ_i denotes the unique nondegenerate n -simplex in the i^{th} copy of $\overline{\Delta}[n+1]$ in $W(n+1)$.

Using the map Ψ_w constructed above for any generator w of $C_n GX$, we define $\Phi_n : C_n GX \rightarrow C_{n+1} GX$ for any 0-reduced simplicial set X by

$$\Phi_n(w) = C_{n+1} \Psi_w \circ \Phi_n(\tau \delta_1^{\alpha_1} \cdot \dots \cdot \tau \delta_k^{\alpha_k}).$$

Note that if $X = W(n+1)$ and $w = \tau \delta_{j_1}^{\alpha_1} \cdot \dots \cdot \tau \delta_{j_k}^{\alpha_k}$, then

$$\Psi_w : GW(n+1) \rightarrow GW(n+1)$$

is a homomorphism of simplicial groups given simply by permuting generators. It follows from the construction of the chain homotopy \tilde{h}^{n+1} and therefore of the chain homotopy \tilde{h}^{n+1} that \tilde{h}^{n+1} is natural with respect to homomorphisms that simply permute generators, so that

$$C_{n+1} \Psi_w \circ \tilde{h}_n^n = \tilde{h}_n^n \circ C_n \Psi_w.$$

Consequently, $\Phi_n(w)$ is indeed well-defined, since ϕ , ψ and, by the induction hypothesis, Φ_{n-1} are all natural with respect to simplicial maps.

Moreover,

$$\begin{aligned} \partial \Phi_n(w) &= C_{n+1} \Psi_w \circ \partial \Phi_n(\tau \delta_1^{\alpha_1} \cdot \dots \cdot \tau \delta_k^{\alpha_k}) \\ &= C_{n+1} \Psi_w \circ ((\phi\psi - Id_{CGW(n)} - \Phi_{n-1}\partial)(\tau \delta_1^{\alpha_1} \cdot \dots \cdot \tau \delta_k^{\alpha_k})) \\ &\stackrel{(\star)}{=} (\phi\psi - Id_{CGX} - \Phi_{n-1}\partial) \circ C_n \Psi_w(\tau \delta_1^{\alpha_1} \cdot \dots \cdot \tau \delta_k^{\alpha_k}) \\ &= (\phi\psi - Id_{CGX} - \Phi_{n-1}\partial)(w), \end{aligned}$$

where the equality (\star) follows from naturality of ϕ , ψ and Φ_{n-1} . In other words,

$$\partial \Phi_n + \Phi_{n-1} \partial = \phi\psi - Id_{CGX},$$

for all X , i.e., condition (2.n) holds.

To conclude, observe that condition (1.n) holds as well, since for all simplicial maps $g : X \rightarrow Y$ between 0-reduced spaces and all $w \in GX$,

$$Gg \circ \Psi_w = \Psi_{Gg(w)} : GW(n+1) \rightarrow GY.$$

□

Remark 16. *It is in order to be able to apply the chain homotopy of Morace and Prouté that we work with 0-reduced simplicial sets. There is no such chain homotopy in the 1-reduced case, so it seems we are obliged to prove the existence of the strong deformation retract in the 0-reduced case in order to conclude that it exists in the 1-reduced case as well.*

Remark 17. *As defined in the proof above, Φ_n is almost certainly not a derivation homotopy, since, as easy computations show, \tilde{h} is not a derivation homotopy.*

Remark 18. Morace and Prouté showed that $\bar{h}_{i+1}^n \circ \bar{h}_i^n = 0$ for all i and n , from which it follows that $\tilde{h}_{i+1}^n \circ \tilde{h}_i^n = 0$ as well and therefore that

$$\tilde{h}_{n+1}^{n+1} \circ \Phi_n(\tau\delta_{j_1}^{\alpha_1} \cdot \dots \cdot \tau\delta_{j_k}^{\alpha_k}) = 0$$

for all j_1, \dots, j_k and $\alpha_1, \dots, \alpha_k$.

Remark 19. The results in this paper generalise from chain complexes to crossed complexes. There is a crossed cobar construction $\Omega\pi X$ on the fundamental crossed complex πX , see [2], and we may define a ‘crossed’ Szczarba map of crossed chain algebras $\phi : \Omega\pi X \rightarrow \pi GX$ that forms part of a deformation retraction

$$\Omega\pi X \begin{array}{c} \xrightarrow{\phi} \\ \xleftarrow{\psi} \end{array} \pi GX \quad \begin{array}{c} \curvearrowright \\ \downarrow \\ \Phi \end{array}$$

The classical argument that ϕ is a weak equivalence, using (12), does not go through in this slightly non-abelian situation, since it seems there is no good notion of twisted tensor product of crossed complexes.

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REFERENCES

1. J. F. Adams, *On the cobar construction*, Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 409–412. MR MR0079266 (18,59c)
2. Hans-Joachim Baues and Andrew Tonks, *On the twisted cobar construction*, Math. Proc. Cambridge Philos. Soc. **121** (1997), no. 2, 229–245. MR MR1426521 (97k:55023)
3. V. K. A. M. Gugenheim and H. J. Munkholm, *On the extended functoriality of Tor and Cotor*, J. Pure Appl. Algebra **4** (1974), 9–29. MR MR0347946 (50 #445)
4. V.K.A.M. Gugenheim, L. Lambe, and J. Stasheff, *Perturbation theory in differential homological algebra*, Ill. J. Math. **33** (1989), 566–582.
5. ———, *Perturbation theory in differential homological algebra ii*, Ill. J. Math. **35** (1991), 357–373.
6. Kathryn Hess, Paul-Eugène Parent, and Jonathan Scott, *A chain coalgebra model for the James map*, Homology, Homotopy Appl. **9** (2007), no. 2, 209–231. MR MR2366950 (2008k:55020)
7. Daniel M. Kan, *Abstract homotopy. IV*, Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 542–544. MR MR0087938 (19,440b)
8. L. Lambe and J. Stasheff, *Applications of perturbation theory to iterated fibrations*, Manuscr. Math. **58** (1987), 363–376.
9. J. Peter May, *Simplicial objects in algebraic topology*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1992, Reprint of the 1967 original. MR MR1206474 (93m:55025)
10. Frédéric Morace and Alain Prouté, *Brown’s natural twisting cochain and the Eilenberg-Mac Lane transformation*, J. Pure Appl. Algebra **97** (1994), no. 1, 81–89. MR MR1310750 (96a:55029)
11. R. H. Szczarba, *The homology of twisted cartesian products*, Trans. Amer. Math. Soc. **100** (1961), 197–216. MR MR0137111 (25 #567)

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